Quasi-periodic bifurcations in a strong resonance

Hoo, Jun

Summary

The dynamics of conservative mechanics are modelled by Hamiltonian systems. These are called integrable if they have sufficient many conserved quantities or constants of motion, this implies a certain toroidal symmetry. Classical examples of integrable Hamiltonian systems are all one-degree-of-freedom systems, the spherical pendulum, the central motion (Kepler problem), the Euler top and the Lagrange top. The latter serves as a leading example in our research.

In this thesis, we consider families of integrable Hamiltonian systems, i.e. dynamical systems depending on parameters as well as their Hamiltonian perturbations. In the product of the phase and parameter spaces, an integrable Hamiltonian family has an invariant submanifold filled up by invariant tori with quasi-periodic motions. Although integrable systems are exceptional in practice, one deals often with non-integrable ones which are good integrable-approximations. One general classical question is whether the quasi-periodic motions in these integrable-approximations can be continued to the original 'nearly-integrable' systems. This persistence problem is the main issue of the so-called Kolmogorov-Arnold-Moser (or KAM) theory.

Suppose that the invariant tori are Lagrangian — in this case their dimension is equal to the number of degrees of freedom — then the classical KAM theory (1954) is applicable. This asserts that for most integrable Hamiltonian systems we have the following: each nearly-integrable Hamiltonian perturbation has a union of invariant Lagrangian tori with quasi-periodic motions. This union, a so-called Cantor family of tori, is nowhere dense and has a large relative measure. Roughly speaking, this means that the majority of the invariant Lagrangian tori in the original integrable system survives small perturbations. A similar stability result also holds for the cases where the invariant torus family consists of isotropic normally elliptic or normally hyperbolic tori. For brevity, all these results are referred to as the 'stan-

dard' KAM theory. We observe that the persistence of torus families with a normally resonant torus does not belong to the domain of this theory. The ultimate goal of the first part of this thesis is to extend the standard KAM theory to this normally resonant case. The generalized results are expressed in terms of *normal linear stability.* These stability results are also applicable to other structure-preserving cases, for instance, to the classes of volume-preserving and reversible dynamical systems. Additional to the Hamiltonian case, we explicitly discuss the general (dissipative) case.

The most interesting normally resonant case is the normal $1 : -1$ resonance. This generically gives rise to a *quasi-periodic Hamiltonian Hopf bifurcation.* The Hamiltonian Hopf bifurcation occurs in many dynamical systems from conservative mechanics: the Lagrange top near gyroscopic stabilisation, the double spherical pendulum near the unstable equilibrium, the restricted three-body problem, the hydrogen atom in crossed electrical and magnetical fields, etc. The bifurcation becomes 'quasi-periodic' when an external or parametrical (quasi-)periodic forcing is applied. The study of this bifurcation is the central item of the second part of this thesis, where the Lagrange top with quasi-periodic forcing is extensively discussed. A quasi-periodic Hamiltonian Hopf bifurcation occurs when the stability-type — normally elliptic or normally hyperbolic — of the invariant tori changes as the parameter varies and a certain generic condition on higher order terms is satisfied. In particular the normally $1 : -1$ resonant torus appears at the center of the bifurcation. If there are $m + 2$ degrees of freedom, then for a suitable integrable-approximation the phase space near this torus is split into invariant m-, $(m + 1)$- and $(m + 2)$-tori. This singular foliation gives a local stratification in a three-dimensional parameter space, where the strata are determined by the dimension of tori. The geometry of this stratification is described by a piece of the swallowtail catastrophe set from singularity theory: the m-tori correspond to the (one-dimensional) singular part of the swallowtail, the elliptic $(m + 1)$-tori to the regular surface and the Lagrangian tori to the open region above the surface. The bifurcation gives rise to a non-trivial monodromy in the family of Lagrangian tori. The original problem now becomes a nearly-integrable perturbation and we are interested in the persistence of the invariant tori from the singular foliation. Combining standard KAM theory with the earlier mentioned normal linear stability, we conclude that the isotropic tori, near the normally resonant torus, of all dimensions survive in Cantor families. Moreover a global KAM theory gives a proper extension of the non-trivial monodromy in the integrable Lagrangian torus bundle to the non-integrable case. Monodromy in the integrable case is important for semi-classical approximations: there is a connection with so-called spectral defects. We expect that monodromy in the nearly-integrable cases will also play an important role.